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PHASE PATTERN CLASSIFICATION FOR THE PROBLEM OF THE MOTION OF A BODY IN A RESISTING MEDIUM IN THE PRESENCE OF A LINEAR DAMPING MOMENT†

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A qualitative analysis is performed for a dynamical system simulating the plane-parallel motion of a body through a medium in jet or separated flow, such that all the interactions of the medium with the body are concentrated on a part of the body having the form of a flat plate [1, 2]. The interaction force is directed along a normal to the plate, and the point of application of this force depends only on the angle of attack. The propulsive force acts along the central perpendicular to the plate and ensures that the value of the velocity of the centre of the plate remains constant throughout the motion. Moreover, a damping moment linear in the angular velocity is imposed on the body, and a classification of the phase patterns of the system is obtained depending on the coefficient of that moment. A mechanical and topological analogy with an anchored pendulum in a flowing medium is pointed out [3, 4].

THE WELL-KNOWN Kirchhoff problem, and also the effects discussed in [5], do not exhaust all possibilities for the motion of a body in a medium when the translational motion of the former is coupled to its rotation. In this paper, we highlight the problem of the motion of a body under the condition that the line of action of the force applied to the body does not change its orientation with respect to the body, but can only be displaced parallel to it depending on the angle of attack. Similar conditions occur for motions with large angles of attack in media with jet [1, 2] or separated [6] flow.

1. STATEMENT OF THE PROBLEM

We will consider a model for the problem of the plane-parallel motion of a rigid body through a resisting medium. It is assumed that the medium does not act on the entire surface of the body, but only on the part that has the shape of a flat plate. Here the interaction of the body with the medium occurs under conditions of jet or separated flow. The latter conditions are satisfied, for example, by bodies of cylindrical shape entering water, by a parachute moving through the air, etc.

The force $\mathbf{S}$ of the interaction of the body with the medium is directed along the normal to the flat plate AB (Fig. 1). The point of application N of this force is determined solely by the angle of attack α, which is measured anticlockwise from the velocity vector v at the centre of the plate D to the normal at this point (the line CD) (so that $DN = y(\alpha)$). We take the value of the resistance force in the form $S = s(\alpha)v^2$, where v is the absolute value of the velocity of the centre of the plate.

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We will take the mass distribution to be such that the centre of mass C of the body is situated at the central perpendicular CD to the plate.

We assume that an additional propulsive force of $\mathbf{T}$ acts on the body along the line CD. The introduction of this force is used both for methodological purposes and to ensure certain specified classes of motion.

To describe the position of the body we shall choose Cartesian coordinates x, y for the centre of the plate and the angle of inclination φ which we will measure clockwise in the plane from the line CD to the axis of the inertial system of coordinates. Here the system phase space (x, y, $\varphi^{\cdot}$, x, y, φ) is six-dimensional. The quantities v and α are expressed by non-integrable relations in terms of the variables x, y and φ. Because of this we specify the phase state of the system by the quantities (v, α, $\varphi^{\cdot}$, x, y, φ) and consider the functions v and α to be quasi-velocities of the system.

Because the generalized forces, and also the kinetic energy, do not depend on the position of the body on the plane, the coordinates (x, y, φ) are cyclic. This enables us to consider a system of ordinary differential equations of lower dimension, the first two equations of which describe the motion of the centre of mass and the last one describes the time variation of the moment of momentum about the Koenig axis

$$\begin{aligned} &v^{\cdot}\cos\alpha - \alpha^{\cdot} v\sin\alpha + \omega v\sin\alpha + \sigma\omega^2 = (T - s(\alpha)v^2)/m \\ &v^{\cdot}\sin\alpha + \alpha^{\cdot} v\cos\alpha - \omega v\cos\alpha + \sigma\omega^{\cdot} = 0 \\ &I\omega^{\cdot} = -y(\alpha)s(\alpha)v^2 \end{aligned} \tag{1.1}$$

(σ is the distance DC, m is the mass of the body, and I is its central moment of inertia).

The following three kinematic relations complete system (1.1) to the sixth order

$$\varphi^{\cdot} = \omega, \quad x^{\cdot} = v\cos(\alpha - \varphi), \quad y^{\cdot} = v\sin(\alpha - \varphi) \tag{1.2}$$

The functions y and s occur in the dynamical system (1.1). To describe them qualitatively we use experimental information on the properties of jet flow.

We shall assume that y is a sufficiently smooth, odd, 2π-periodic function satisfying the following conditions: $y(\alpha) > 0$ when $\alpha \in (0, \pi)$, with $y'(0) > 0$, $y'(\pi/2)$ and also $y(\alpha+\pi) = -y(\alpha)$.

We shall also assume that s is a sufficiently smooth, even, 2π-periodic function satisfying the following conditions: $s(\alpha) > 0$ when $\alpha \in (0, \pi/2)$, $s(\alpha) > 0$, when $\alpha \in (\pi/2, \pi)$, with $s(0) > 0$, $s'(\pi/2) < 0$, and also $s(\alpha+\pi) = -s(\alpha)$. We shall denote the class of such functions s by Σ.

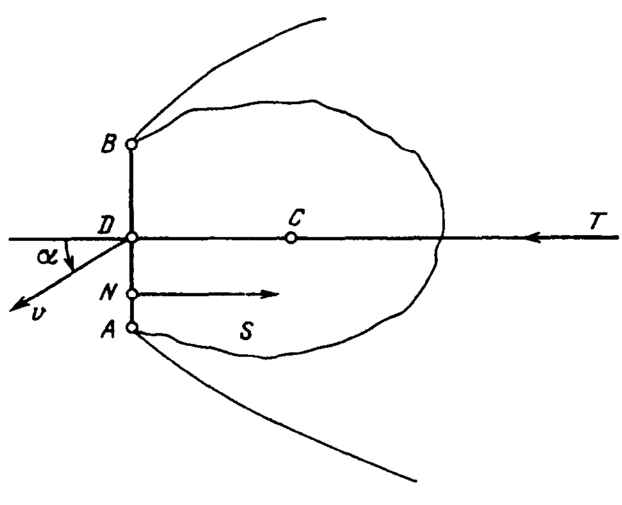

FIG. 1.

We will consider the product $F(\alpha) = y(\alpha)s(\alpha)$. A consequence of the preceding definitions is the assertion that F is a sufficiently smooth, odd, π-periodic function satisfying the following conditions: $F(\alpha) > 0$ when $\alpha \in (0, \pi/2)$, with $F'(0) > 0$, $F'(\pi/2) < 0$. The class of such functions F will be denoted by Φ.

Analysis of the resulting dynamical systems leads to the discovery of mechanical and topological analogies. The dynamical system obtained describes: (a) a free body in a resisting medium, with the velocity of the centre of the plate being constant, and (b) an anchored pendulum in a flowing medium.

Here the equation for the angle of attack of the free body is equivalent to the equation for the angle of displacement of the anchored pendulum, and the constant velocity of the centre of the plate of the free body corresponds to the constant velocity of the flow incident on the anchored pendulum [3, 4, 7–10, 11].

2. ON STRUCTURAL STABILITY

By a structurally stable (coarse) system of differential equations we mean a system such that under small deformations that belong not to the entire class of functions, but only to some subclass, it is equivalent to the original system [12, 13].

The theory of the dynamical system (1.1) involves the following aspects: investigation of the topological structures of two- and three-dimensional phase patterns, and also the proof of structural stability (robustness) of these dynamical systems depending on the value of the controlling propulsion [4, 7–11].

We shall consider a deformation of the right-hand side of the system that belong not to the entire class of right-hand sides, but to the subclass

$$K \subset C^r, \quad r \geqslant 1$$

Definition. A system of differential equations specifying a sufficiently smooth vector field $\mathbf{V}$ will be referred to as structurally stable (coarse) with respect to the class of functions $K \subset C^r$ (relatively structurally stable) if any vector field W defined with the aid of the class of functions K and obtained by deforming the field $\mathbf{V}$ in the C^1 topology relative to the class of functions K is topologically equivalent to the field $\mathbf{V}$.

For some fixed functions T system (1.1) is structuraly stable (coarse) with respect to the classes Φ and Σ. Furthermore, any two vector fields defined with the aid of the classes Φ and Σ are topologically equivalent.

The following conditions are satisfied

$$F = F_0(\alpha) = AB\sin\alpha\cos\alpha \in \Phi, \quad s = s_0(\alpha) = B\cos\alpha \in \Sigma \tag{2.1}$$

If there is structural stability for system (1.1) for some function T with respect to the classes Φ and Σ then when considering system (1.1) one can restrict oneself to an analytic system of equations (1.1) with conditions (2.1).

It is known that coarse systems are not dense in the C^1 topology [14]. But if one considers coarse systems with respect to the subclass $K \subset C^r$, then it may turn out that in the C^1 topology the given systems can generate an everywhere dense set. Indeed, for certain fixed functions T coarse systems (1.1), defined with respect to classes Φ and Σ, are everywhere dense in the C^1 topology.

3. A SYSTEM OF DIFFERENTIAL EQUATIONS

We consider the motions of a body such that at all times the magnitude of the velocity of the centre of the plate is constant. Here we impose a non-holonomic constraint on the system, and the force T is the reaction of this non-holonomic constraint [9, 10].† The theory for this case has been developed in [4, 7–11].‡

On the other hand, the equation $v(t) = \text{const}$ specifies in the phase space a class of level surfaces of the first integral. Here the function T is the control supporting the motion of the system on a given surface.

Thus, an independent second-order subsystem decouples from system (1.1). Inside and only inside the manifold

$$O = \{ (\alpha, \omega) \in R^2 : \cos\alpha = 0 \}$$

this subsystem is equivalent to the following system with $h = 0$

$$\begin{aligned} &\alpha^{\cdot} = \omega + A_1 F(\alpha)/\cos\alpha, \quad \omega^{\cdot} = A_2 F(\alpha) - h\omega \\ &A_1 = \sigma v/I, \quad A_2 = -v^2/I \end{aligned} \tag{3.1}$$

For the details of reducing system (1.1) to system (3.1) see [7, 8] and the author's dissertation.

We shall assume that a damping moment linear in the angular velocity is imposed on the body, and that it only changes the angular acceleration. Thus the change in the angle of attack and the angular velocity is given by system (3.1).

Under condition (2.1), system (3.1) assumes the form of the analytic system

$$\begin{aligned} &\alpha^{\cdot} = \omega + A_1 \sin\alpha, \quad \omega^{\cdot} = A_2 \sin\alpha\cos\alpha - h\omega \\ &A_1 = \sigma v AB/I, \quad A_2 = -v^2 AB/I \end{aligned} \tag{3.2}$$

Below we consider system (3.2) because the general system (3.1) is topologically equivalent to system (3.2) either with respect to the class Φ or with respect to the space of functions C^1. Moreover, system (3.1) is relatively structurally stable for the class Φ.

4. QUALITATIVE ANALYSIS OF THE DYNAMICAL SYSTEM

Classification of singular points. System (3.2) (like system (3.1)) has stationary points $(k\pi, 0)$, and also

$$(\arccos(-hA_1/A_2), \ -A_1 \sin\arccos(-hA_1/A_2)) \tag{4.1}$$

which exist only when

$$A_2/A_1 < h < -A_2/A_1 \tag{4.2}$$

†See also: EROSHIN V. A., SAMSONOV V. A. and SHAMOLIN M. V., On the motion of a body in a medium with jet flow. In *Proceedings of the All-Union Meeting on the Stability of Motion, Oscillations of Mechanical Systems and Aerodynamics* (2-4II. 1988). Deposited in VINITI 22.12.88, No. 8886-B-88.

‡See also: SHAMOLIN M. V., Qualitative analysis of a model problem on the motion of a body in a medium with jet flow. Candidate dissertation, Moscow State University, Moscow, 1991.

Here and throughout $k=0, \pm 1, \pm 2, \ldots$. The roots of the characteristic equation of system (3.2) near the points $(2k\pi, 0)$ have the form

$$\lambda_{1,2} = \tfrac{1}{2}(A_1 - h \pm [(h + A_1)^2 + 4A_2]^{1/2})$$

and the bifurcation set consists of the following constants

$$\{ A_1, -A_2/A_1, -A_1 + 2\sqrt{-A_2}, -A_1 - 2\sqrt{-A_2} \}$$

For the points $((2k+1)\pi, 0)$ we have similar expressions with A_1 replaced by $-A_1$.

If they exist, the stationary points (4.1) are of saddle type. Indeed, the points of the field (4.1) only exist when condition (4.2) is satisfied, which ensures that the characteristic equation has real roots of opposite signs near the points (4.1).

Below we use for convenience the following notation and abbreviations: $\xi = A_2/A_1$, $\eta_+^\pm = A_1 \pm \sqrt{(-A_2)}$, $\eta_-^\pm = -A_1 \pm \sqrt{(-A_2)}$, S is a saddle, SN is a stable node, UN is an unstable node, SF is a stable focus, UF is an unstable focus, SC is a stable limit cycle and UC is an unstable limit cycle.

The classification of the stationary points (0, 0) and $(\pi, 0)$ is given in Tables 1–4 and corresponds to the following four cases:

Case 1A (Table 1)

$$\xi < \eta_-^- < \eta_+^- < -A_1 < 0 < A_1 < \eta_-^+ < \eta_+^+ < -\xi$$

Case 1B (Table 2)

$$\eta_-^- < \xi < \eta_+^- < -A_1 < 0 < A_1 < \eta_-^+ < -\xi < \eta_+^+$$

Case 2 (Table 3)

$$\eta_-^- < -A_1 < \xi < \eta_+^- < 0 < \eta_-^+ < -\xi < A_1 < \eta_+^+$$

Case 3 (Table 4)

$$\eta_-^- < -A_1 < \eta_-^+ < \xi < 0 < -\xi < \eta_+^- < A_1 < \eta_+^+$$

TABLE 1

Subcase No.	Range of variation of h	Stationary point type (0, 0)	(π, 0)	Figure No.
1	$(-\infty, \xi)$	UN	S	2
2	(ξ, η_-^-)	UN	UN	9*
3	(η_-^-, η_+^-)	UF	UN	9*
4	$(\eta_+^-, -A_1)$	UF	UF	9
5	$(-A_1, 0)$	UF	SF	UC-5 7
6	$(0, A_1)$	UF	SF	SC-4 6
7	(A_1, η_-^+)	SF	SF	8
8	(η_-^+, η_+^+)	SN	SF	8*
9	$(\eta_+^+, -\xi)$	SN	SN	8*
10	$(-\xi, +\infty)$	S	SN	3

TABLE 2

Subcase No.	Range of variation of h	Stationary point type (0, 0)	Stationary point type $(\pi, 0)$	Figure No.
1	$(-\infty,\ \eta_-^-)$	UN	S	2
2	$(\eta_-^-,\ \xi)$	UF	S	2*
3	$(\xi,\ \eta_+^-)$	UF	UN	9*
4	$(\eta_+^-,\ -A_1)$	UF	UF	9
5	$(-A_1,\ 0)$	UF	SF	UC-5 7
6	$(0,\ A_1)$	UF	SF	SC-4 6
7	$(A_1,\ \eta_-^+)$	SF	SF	8
8	$(\eta_-^+,\ -\xi)$	SN	SF	8*
9	$(-\xi,\ \eta_+^+)$	S	SF	3*
10	$(\eta_+^+,\ +\infty)$	S	SN	3

TABLE 3

Subcase No.	Range of variation of h	Stationary point type (0, 0)	Stationary point type $(\pi, 0)$	Figure No.
1	$(-\infty,\ \eta_-^-)$	UN	S	2
2	$(\eta_-^-,\ -A_1)$	UF	S	2*
3	$(-A_1,\ \xi)$	UF	S	2*
4	$(\xi,\ \eta_+^-)$	UF	SN	7*
5	$(\eta_+^-,\ 0)$	UF	SF	7
6	$(0,\ \eta_-^+)$	UF	SF	6
7	$(\eta_-^+,\ -\xi)$	UN	SF	6*
8	$(-\xi,\ A_1)$	S	SF	3*
9	$(A_1,\ \eta_+^+)$	S	SF	3*
10	$(\eta_+^+,\ +\infty)$	S	SN	3

Global qualitative analysis. It follows from the results of the author's dissertation that for a system of the form (3.1), when the condition $F \in \Phi$ is satisfied for any $h \in R$, a phase trajectory that has horizontal and vertical asymptotes in the phase plane in the neighbourhood of the point at infinity does not exist. As will be clear from the phase patterns, under certain conditions for a system of the form (3.1) inclined asymptotes to certain trajectories in the neighbourhood of the point at infinity exist, which is then either attractive or repulsive.

The question of the existence of closed trajectories that can be contracted to a point along the phase cylinder reduces to investigating the existence of such trajectories around either the points $(2k\pi,\ 0)$, $k \in Z$, or $((2k+1)\pi,\ 0)$ [11]. Here in the strip $\Pi = \{(\alpha,\ \omega) \in R^2:\ -\pi/2 < \alpha < \pi/2\}$ there sometimes exists a unique stable limit cycle, while in the strip $\Pi' = \{(\alpha,\ \omega) \in R^2:\ \pi/2 < \alpha < 3\pi/2\}$ a unique unstable limit cycle exists. (In the first case, the size of the cycle decreases as h increases, and in the second one it increases.)

Remarks. 1. There exists an $h_1^* > 0$ (respectively $h_2^* < 0$) such that when $h \in (-\infty,\ h_1^*) \cup (A_1F'(0),\ +\infty)$ (respectively $h \in (-\infty,\ -A_1F'(0)) \cup (h_2^*,\ +\infty)$) in the strip Π (respectively Π') there are no closed trajectories or closed curves composed of trajectories. For systems of the form (3.2) we have the estimates

$$\frac{2}{\pi}A_1 < h_1^* < A_1, -A_1 < h_2^* < -\frac{2}{\pi}A_1$$

2. The interval for the possible existence of a cycle is even narrower. One can show that a cycle exists only for $h \in (h_1^*, A_1)$, where h_1^* is a non-zero root of the equation

$$\mathrm{tg}\,\epsilon = -A_2 \epsilon / A_1^2$$

And similarly for h_2^*.

3. Along with the central symmetry about the points $(k\pi, 0)$, the vector field of system (3.1) has a certain broadened mirror symmetry, namely the ω-component of the vector field of system (3.1) changes sign when $\pi/2 + \pi k - \alpha$ is replaced by $\pi/2 + \pi k + \alpha$ and h by $-h$, while the α-component remains unchanged.

For $h = 0$ system (3.1) possesses a certain mirror symmetry without h being transformed: in particular, at mirror-symmetric points with respect to the lines

$$\Lambda_i = \{ (\alpha, \omega) \in R^2 : \alpha + \pi/2 + \pi i \}$$

the α-components remain the same while the ω-components change sign.

By modifying the Dulac and Bendixon criteria, it can be shown that closed trajectories that do not contract to a point along the phase cylinder do not exist when $h \neq 0$. At $h = 0$ a continuum of trajectories of this topological type exists [4, 7–11] which forms a comparison system for the dynamical system (3.1) when $h \neq 0$.

The numerous problems of the global behaviour of all the separatrices with hyperbolic saddles are investigated for each phase pattern separately: this is a development of the theory of Poincaré topographic systems and comparison systems. For brevity, we will not describe this investigation here. For every specific topological type of phase pattern the behaviour of the separatrices is shown in Figs 2–9.

4. For cases 1A.1, 1A.10, 1B.1, 1B.2, 1B.9, 1B.10, 2.1, 2.2, 2.3, 2.8, 2.9, 2.10, 3.1, 3.2, 3.3, 3.4, 3.7, 3.8, 3.9, 3.10 and only for these cases there are no equilibrium positions (4.1).

5. The condition $A_1^2 F'(0) + A_2 < 0$ is not only sufficient, but is also necessary. Thus, for cases 1A.1, 1A.2, 1A.3, 1A.4, 1A.7, 1A.8, 1A.9, 1A.10, 1B.1, 1B.2, 1B.3, 1B.4, 1B.7, 1B.8, 1B.9, 1B.10, 2, 3 there are no limit cycles anywhere in the $R^2\{\alpha, \omega\}$ plane.

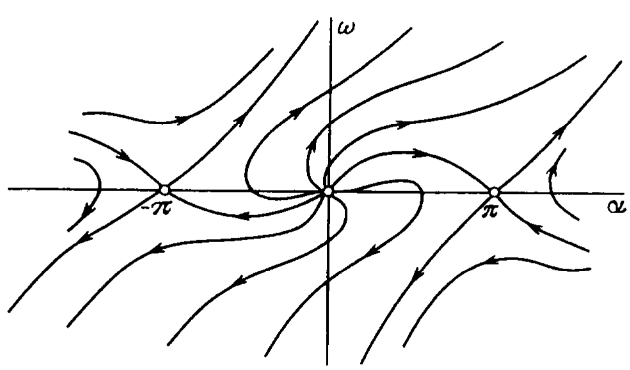

FIG. 2.

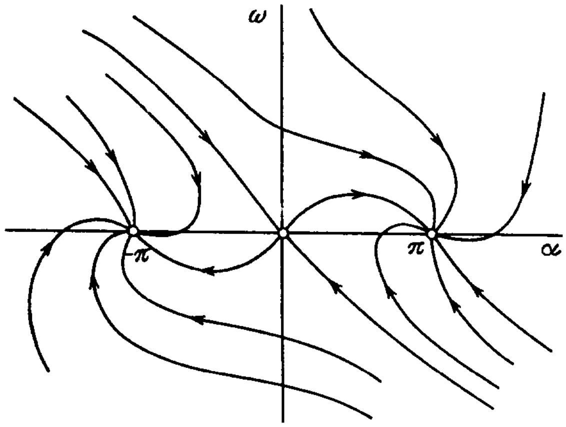

FIG. 3.

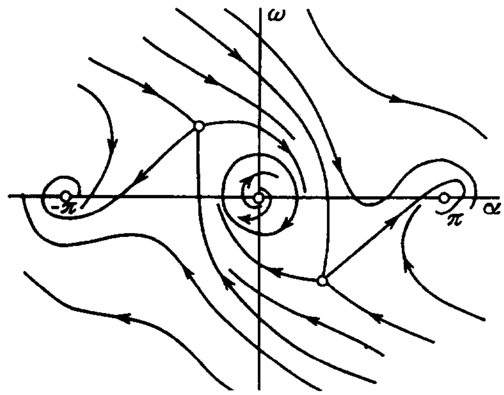

FIG. 4.

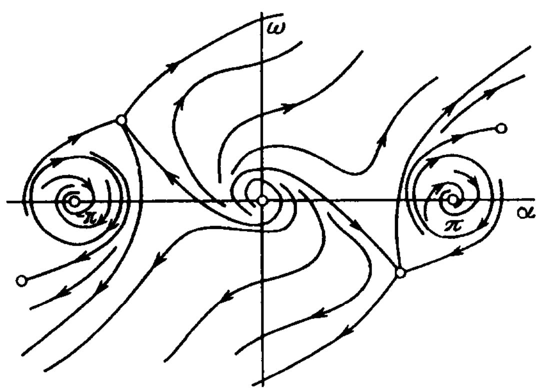

FIG. 5.

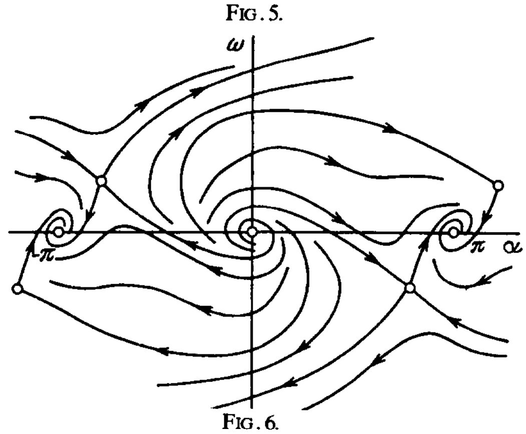

FIG. 6.

TABLE 4

Subcase No.	Range of variation of h	Stationary point type (0, 0)	(π, 0)	Figure No.
1	$(-\infty,\ \eta_-^-)$	UN	S	2
2	$(\eta_-^-,\ -A_1)$	UF	S	2*
3	$(-A_1,\ \eta_-^+)$	UF	S	2*
4	$(\eta_-^+,\ \xi)$	UN	S	2
5	$(\xi, 0)$	UN	SN	7*
6	$(0, -\xi)$	UN	SN	6*
7	$(-\xi,\ \eta_+^-)$	S	SN	3
8	$(\eta_+^-,\ A_1)$	S	SF	3*
9	$(A_1,\ \eta_+^+)$	S	SF	3*
10	$(\eta_+^+, +\infty)$	S	SN	3

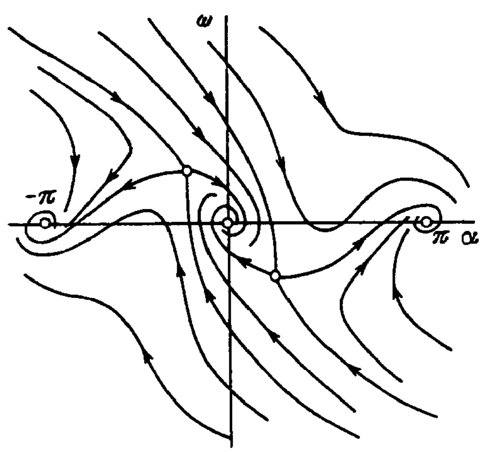

FIG. 7.

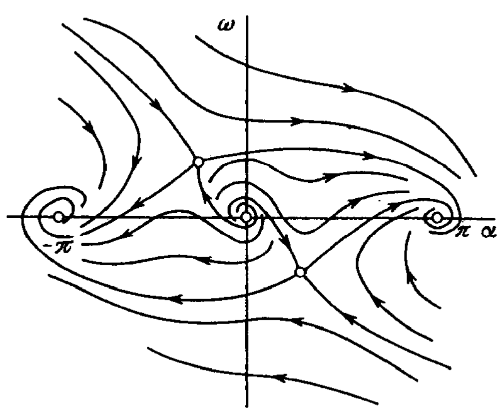

FIG. 8.

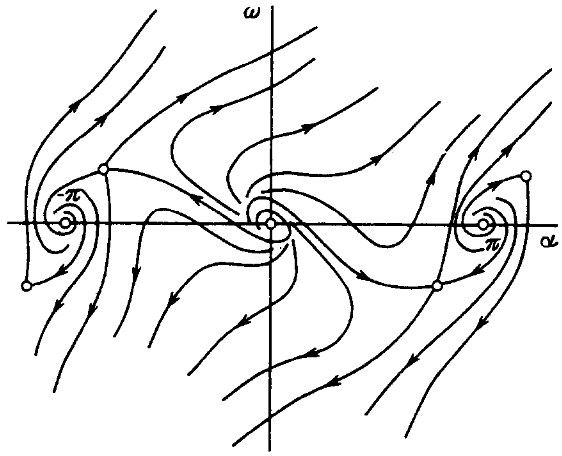

FIG. 9.

Phase pattern classification. Information about the correspondence between the subcases of the four cases 1A, 1B, 2, 3 and the phase patterns is given in Tables 1–4. When there is incomplete correspondence between the results and the indicated phase patterns the figure number is marked by an asterisk. (In place of a node a focus may be shown, and vice versa.) The true types of stationary point is also given in Tables 1–4.

REFERENCES

1. CHAPLYGIN S. A., *Collected Papers.* Nauka, Moscow, 1976.
2. GUREVICH G. I., *Ideal Fluid Jet Theory.* Nauka, Moscow, 1979.
3. LOKSHIN B. Ya., PRIVALOV V. A. and SAMSONSOV V. A., *Introduction to the Motion of a Body through a Resisting Medium.* Izd. MGU, Moscow, 1986.
4. RYZHOVA V. Ye. and SHAMOLIN M. V., Some analogies in the problem of the motion of a body through a resisting medium. In *Proceedings of the VIII All-Union Meeting on Theoretical and Applied Mechanics*, Inst. Probl. Mekh. Akad. Nauk SSSR, Moscow, 1991.
5. MOUILLARD L. P., *L'empire de l'Air.* Paris, 1881.
6. TABACHNIKOV V. G., Steady-state characteristics of wings at low velocities for all angles of attack. *Tr. TsAGI* **1621,** 79–93, 1974.
7. SAMSONOV V. A. and SHAMOLIN M. V., On the problem of the motion of a body through a resisting medium. *Vestn. MGU Ser. 1, Matem. Mekh.* **3,** 51–54, 1989.
8. SAMSONOV V. A. and SHAMOLIN M. V., A model of jet flow about a moving body. MGU Inst. Mekh. Report 3969, Moscow, 1990.
9. SAMSONOV V. A. and SHAMOLIN M. V., Motion through a resisting medium. In *Proceedings of the All-Union Meeting on Contemporary Problems in the Mechanics and Technology of Machine Construction*, pp. 128–129, VINITI, Moscow, 1989.
10. SAMSONOV V. A. and SHAMOLIN M. V., A model of jet flow about a moving body. *Proceedings of the Second All-Union Conference on Non-linear Oscillations of Mechanical Systems.* Izd. Gor'kovsk. Univ. Vol. 2, pp. 95–96, 1990.
11. SHAMOLIN M. V., Closed trajectories of different topological types in the problem of the motion of a body through a resisting medium. *Vestn. MGU Ser. 1, Matem. Mekh.* **2,** 52–56, 1992.
12. NITECKI Z., *Differential Dynamics: An Introduction to the Orbit Structure of Diffeomorphisms.* MIT Press, Cambridge, MA, 1971.
13. PALIS J. and DE MELO W., *Geometric Theory of Dynamical Systems: An Introduction.* Springer, New York, 1982.
14. SMALE S., Coarse systems are not dense. *Period. Sb. Perev. Inostr. Statei* **11,** 4, 107–112, 1967.

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